

ENGINEERING SURVEYING 1

EARTHWORKS and VOLUMES

Earthworks are required for many types of construction such as roads and railways, dams, tunnels, buildings, swimming pools and tanks. For many of these construction projects, the surveyor is required to set out the extent of excavation and calculate quantities of material to be exported or imported. Many volumes encountered in engineering surveying appear at first glance to be rather complex in shape but usually they can be split into basic geometric shapes (or solids) – *prisms*, *wedges* and *pyramids* together with *truncated prisms*. The volume of complex figures can then be computed as a sum of volumes of constituent parts. In addition to these basic geometric shapes, many volumes encountered in engineering can be represented as solids known as *prismoids*, a particular type of solid having a relatively simple volume formula. Descriptions and formulae of basic geometrical figures encountered in engineering surveying problems are set out below.

1. BASIC GEOMETRICAL FIGURES

1.1 Prism

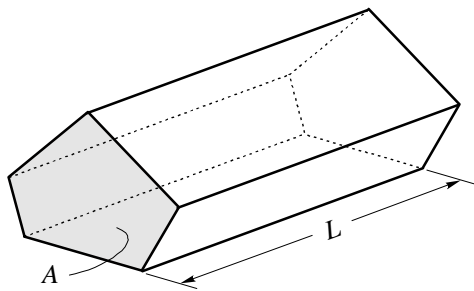


Figure 1.1

This is a solid whose ends are parallel, polygonal and equal in size and shape and whose side faces are parallelograms. If the end faces are perpendicular to the axis of the prism then it is known as a *right prism* and the side faces will be rectangles.

Volume = end area \times perpendicular distance

or

$$V = AL \quad (1.1)$$

1.2 Wedge

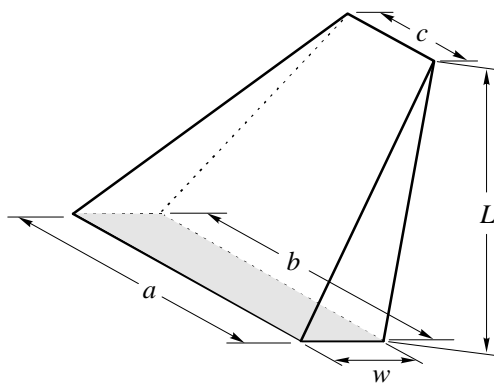


Figure 1.2

This is a solid of five sides; a rectangular base, two rhomboidal sides meeting in an edge and two triangular side faces. The triangular side faces are not necessarily parallel or perpendicular to the rectangular base. The edges a , b and c are parallel.

Volume = sum of parallel edges \times width of base
 $\times \frac{1}{6}$ perpendicular height

or

$$V = \frac{L}{6} w(a + b + c) \quad (1.2)$$

1.3 Pyramid

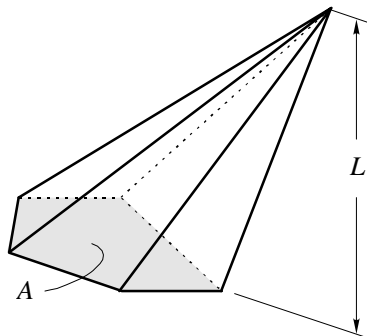


Figure 1.3

A solid figure having a polygonal base, the sides of which form the bases of triangular surfaces meeting at a common vertex.

$$\text{Volume} = \frac{1}{3} \text{ base area} \times \text{perpendicular height}$$

or

$$V = \frac{A}{3} L \quad (1.3)$$

1.4 Truncated Right Triangular Prism

A Truncated Right Triangular Prism is a prism where the end faces ABC (the triangular base) and EHK are not parallel. The side faces $ABEH$, $BCKE$ and $CAHK$ are perpendicular to the base ABC .

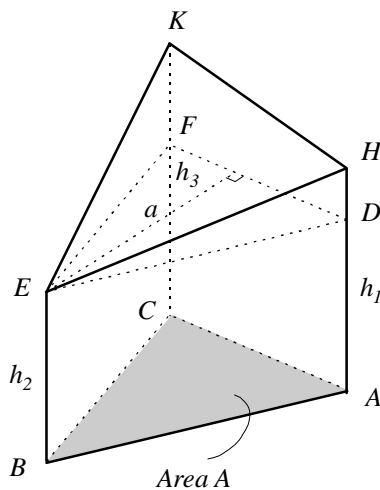


Figure 1.4

Let A be the area of the base ABC (and DEF , a section parallel to the base), h_1 , h_2 , h_3 be the heights AH , BE , CK , and a be the perpendicular height of triangle DEF . The volume of the *Truncated Right Triangular Prism* is the sum of the volume of the prism $ABCDEF$ plus the volume of the pyramid $EDHKF$.

$$V(\text{prism}) = A \times AD = A \left(\frac{3AD}{3} \right) = A \left(\frac{AD + BE + CF}{3} \right)$$

$$V(\text{pyramid}) = \frac{a}{3} (\text{area } DFKH) = \frac{a}{3} \left(\frac{KF + HD}{2} \right) FD$$

$$\begin{aligned} V &= V(\text{prism}) + V(\text{pyramid}) \\ &= \frac{A}{3} (KF + HD + AD + BE + CF) \end{aligned}$$

giving

$$V = \frac{A}{3} (h_1 + h_2 + h_3) \quad (1.4)$$

1.5 Truncated Right Rectangular Prism

A Truncated Right Rectangular Prism is a prism where the end faces $ABCD$ (the rectangular base) and $EFGH$ are not parallel. The side faces are perpendicular to the base.

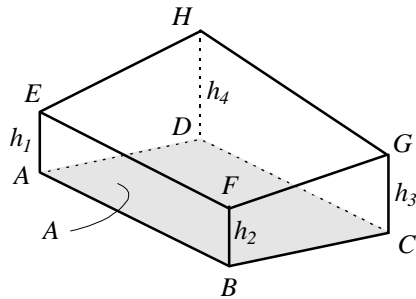


Figure 1.5

In a similar manner to the Truncated Right Triangular prism, the volume of the *Truncated Right Rectangular Prism* is given by

$$V = \frac{A}{4}(h_1 + h_2 + h_3 + h_4) \quad (1.5)$$

1.6 Prismoid

A *prismoid* is a solid figure having parallel end faces, not necessarily similar or having the same number of sides, and having side faces which are plane figures (parallelograms, rectangles, trapezia, triangles etc) extending the full length of the solid.

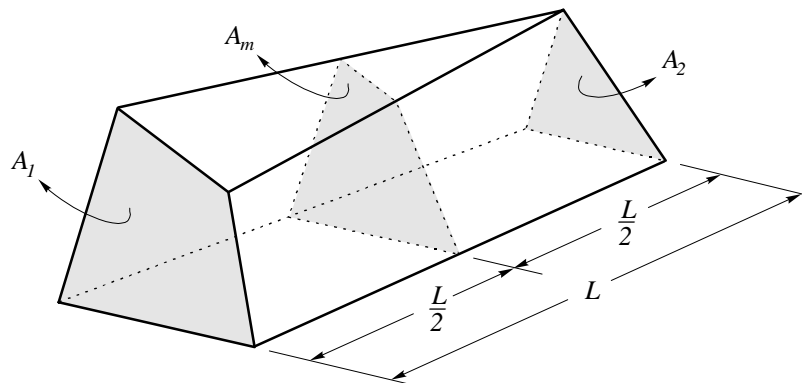


Figure 1.6

A prismoid is shown in Figure 1.6. L is the perpendicular distance between end faces, A_1 , A_2 the areas of the end faces and A_m is the area of the mid-section. Note that the mid-section is parallel with the end faces but its area is not necessarily the mean of A_1 and A_2 . The volume is given by the *Prismoidal Formula*

$$V = \frac{L}{6}(A_1 + 4A_m + A_2) \quad (1.6)$$

A proof of the *Prismoidal Formula* (1.6) is given below.

1.7 Newton's Proof of the Prismoid Formula

A prismoid can be broken into prisms, wedges and pyramids, and the usual proofs of the Prismoidal Formula use this property. The Prismoidal Formula is a computational formula dating from antiquity and appears on one of the oldest documents in existence, a papyrus roll (about 544 centimetres long and 8 centimetres wide), written in Egypt around 1890BC. This papyrus roll commonly known as the Moscow Papyrus (or Golenischev Papyrus after the Russian who purchased it in Egypt in 1893 and brought it to Moscow, where it still resides), contains 25 mathematical problems with solutions. The 14th problem asks for the volume of a truncated pyramid (frustum) and its stated solution can be expressed in the common form we know as the Prismoidal Formula.

The proof of the Prismoidal Formula set out below, was enunciated by Sir Isaac Newton (1642-1726) and can be found in *Plane and Geodetic Surveying*, 5th edn, by D. Clark, Constable & Co., London, 1957. It is interesting to note that that Newton held the view (outlandish at the time) that he and others were just re-discovering the knowledge of the ancient Egyptians. Interesting historical information regarding the Prismoidal Formula (Moscow Papyrus) can be found at <http://www.mathpages.com/home/kmath189/kmath189.html>. For those interested in the history of mathematics, the 1st volume of *The World of Mathematics* by James R. Newman (Simon & Schuster, New York, 1956) has a wonderful description of the *Rhind Papyrus*; another ancient Egyptian scroll describing fundamental mathematical principles.

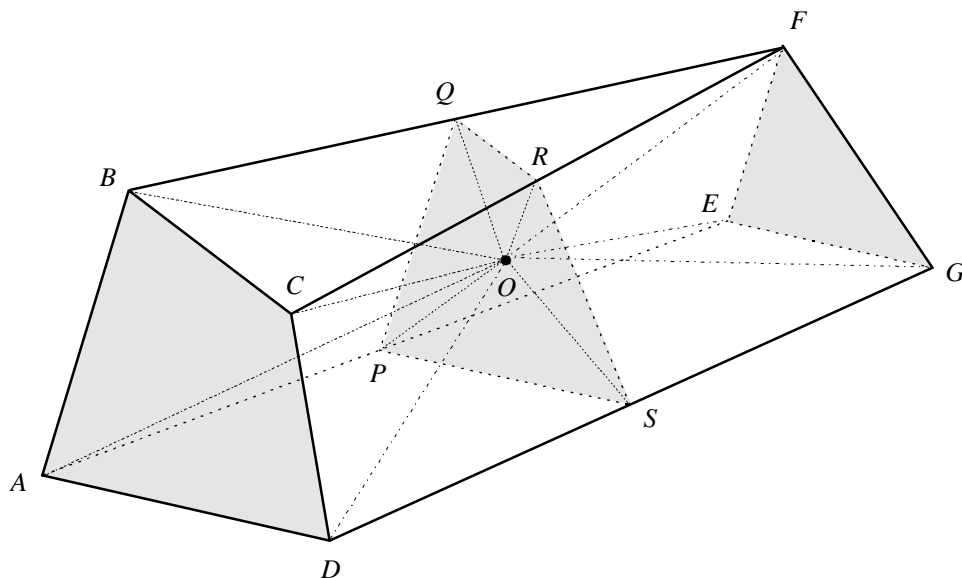


Figure 1.7

A prismoid is shown in Figure 1.7. Let $PQRS$ represent the section of area A_m midway between the end faces $ABCD$ and EFG and parallel to them. Take any point O in the plane of the mid-section and join O to the vertices of both end polygons. The prismoid is thus divided into a number of pyramids, each having its apex at O , and the bases of these pyramids form the end and side faces of the prismoid. Denoting the end areas by A_1 and A_2 and the length of the prismoid by L , the volume of the pyramids based on the end faces are, respectively

$$\frac{A_1}{3} \times \frac{L}{2} = \frac{L}{6} A_1 \quad \text{and} \quad \frac{A_2}{3} \times \frac{L}{2} = \frac{L}{6} A_2$$

To express the volume of the pyramids based on the side faces of the prismoid, consider, say, pyramid $OADGE$, and let the perpendicular distance of O from SP be h , then the volume of the pyramid $OADGE$ is

$$\frac{1}{3} \text{ area}(ADGE) \times h = \frac{1}{3} (PS \times L \times h) = \frac{L}{3} \times 2 \Delta OPS$$

where $2 \Delta OPS$ means $2 \times \text{area of triangle } OPS$.

In the same manner, the volume of pyramid $OCDGF = \frac{L}{3} \times 2\Delta ORS$ and so on for the others, so that the volume of the prismoid is given by

$$\begin{aligned} V &= \frac{L}{6}A_1 + \frac{L}{6}A_2 + \frac{2L}{3}(\Delta OPS + \Delta ORS + \Delta OQR + \Delta OPQ) \\ &= \frac{L}{6}A_1 + \frac{L}{6}A_2 + \frac{L}{6}(4A_m) \\ &= \frac{L}{6}(A_1 + 4A_m + A_2) \end{aligned}$$

2. VOLUME COMPUTATION

2.1 Volume by End Area Formula

For many volume computations, the solid, whose faces are planes extending the whole length of the solid, has end faces, which are vertical planes. Such figures arise in earthworks for roads where the end faces are vertical sections at regular chainages along the centre line. Figure 2.1 shows such a solid.

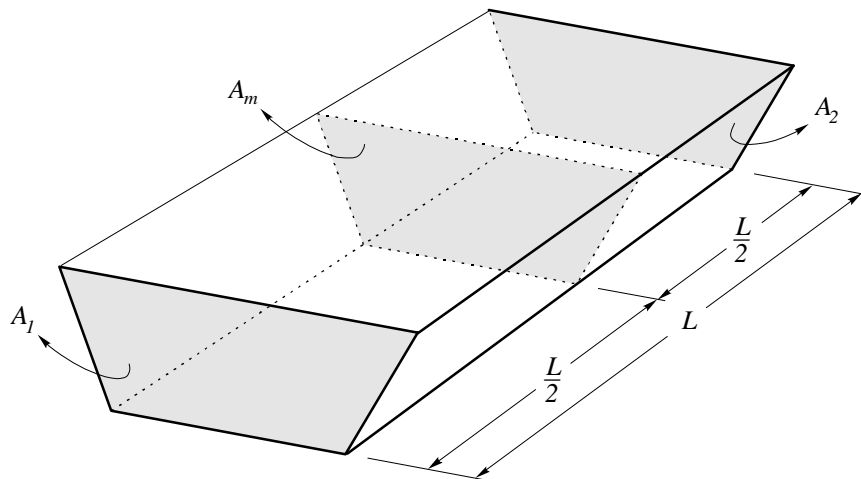


Figure 2.1

If A_1 and A_2 are the areas of two cross-sections distance L apart, then the volume between the two is given by the *End Area Formula*

$$V = \left(\frac{A_1 + A_2}{2} \right) L \quad (2.1)$$

For n sections, each a distance L apart then the total volume is given by

$$V = \left(\frac{A_1 + A_n}{2} + A_2 + A_3 + \cdots + A_{n-1} \right) L \quad (2.2)$$

The End Area Formula is valid if the mid-section (unknown) is the mean of the end areas (known). This will be true if the solid is composed of *prisms* and *wedges*, but is not so if the solid contains any *pyramids*.

To explain this consider the volume of a pyramid, $V = \frac{L}{3} A$, see equation (1.3).

Using the *End Area Formula* the volume of a pyramid is $V = \left(\frac{A_1 + A_2}{2} \right) L = \frac{L}{2} A_1$ since $A_2 = 0$.

The error in the volume is $V_{error} = \frac{L}{2} A_1 - \frac{L}{3} A_1 = \frac{L}{6} A_1$, ie, half the volume of the pyramid. Hence, the *End Area Formula* will overestimate the volume of a prismoid by half the volume of any pyramids contained within the prismoid.

2.2 Comparison of End Area and Prismoidal Formulae

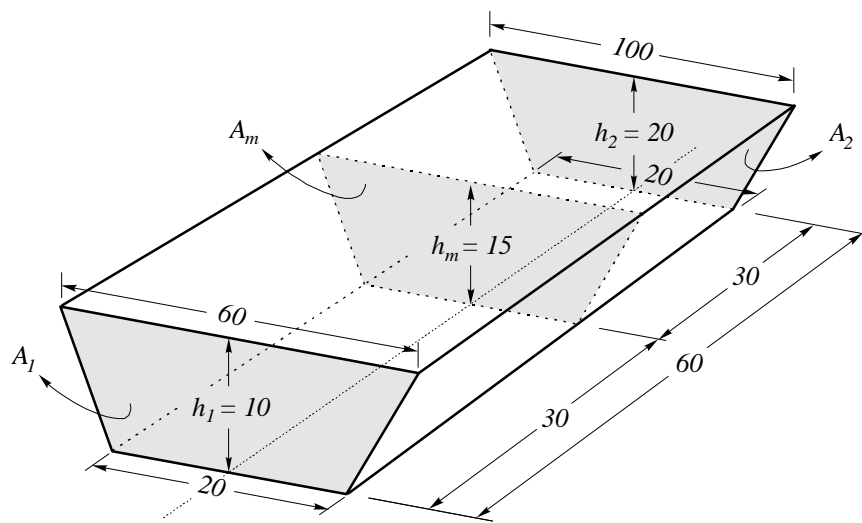


Figure 2.2

Figure 2.2 shows a prismoid where the end faces are vertical planes and the side faces are batter planes of slope 1 in 2 (1 vertical to 2 horizontal). The base of the prismoid is a horizontal plane and the top is a sloping plane. The top and bottom of the vertical cross sections are level lines.

The areas of the cross sections are: $A_1 = \left(\frac{60 + 20}{2} \right) 10 = 400 \text{ m}^2$, $A_m = 750 \text{ m}^2$ and $A_2 = 1200 \text{ m}^2$

Volume by End Area Formula:

$$\begin{aligned} V &= \left(\frac{A_1 + A_2}{2} \right) L \\ &= \left(\frac{400 + 1200}{2} \right) 60 \\ &= 48000 \text{ m}^3 \end{aligned}$$

Volume by Prismoidal Formula

$$\begin{aligned} V &= \frac{L}{6} (A_1 + 4A_m + A_2) \\ &= \frac{60}{6} (400 + 4(750) + 1200) \\ &= 46000 \text{ m}^3 \end{aligned}$$

The volume obtained by the Prismoidal Formula, (46000 m^3) is correct and the volume obtained by the End Area Formula is greater by 2000 m^3 ($\simeq 4\%$). This overestimation is the general rule when computing volumes using the End Area Formula. If the section areas are not changing rapidly (ie, $A_1 \simeq A_2$) then the error is usually negligible and is ignored in practice.

2.3 Prismoidal Correction (Prismoidal Excess)

If the volume computed by the End Area Formula is considered to be in excess of the true volume by a significant amount, the true volume can be obtained by applying the *Prismoidal Correction (P.C.)*

$$P.C. = \frac{L}{12} (w_1 - w_2)(c_1 - c_2) \text{ m}^3 \quad (2.3)$$

where w_1 and w_2 are the horizontal components of the natural surface distances between batter slopes and c_1 and c_2 are the centre line depths of cut (or heights of fill) for the cross sections.

Example:

For the prismoid shown in Figure 2.2, the volume by the End Area Formula was 48000 m^3 . With $w_1 = 60$, $w_2 = 100$, $c_1 = 10$, $c_2 = 20$ and $L = 60$. The Prismoidal Correction is

$$P.C. = \frac{60}{12} (60 - 100)(10 - 20) = 2000 \text{ m}^3$$

The true volume is then

$$V = 48000 \text{ m}^3 - 2000 \text{ m}^3 = 46000 \text{ m}^3$$

This is the volume obtained by the Prismoidal Formula.

In practice, this formula (correction) is rarely used and the field procedure is tailored to accord with the End Area method of volume computation.

3. PLANES and BATTER SLOPES

3.1 Equation of a Plane

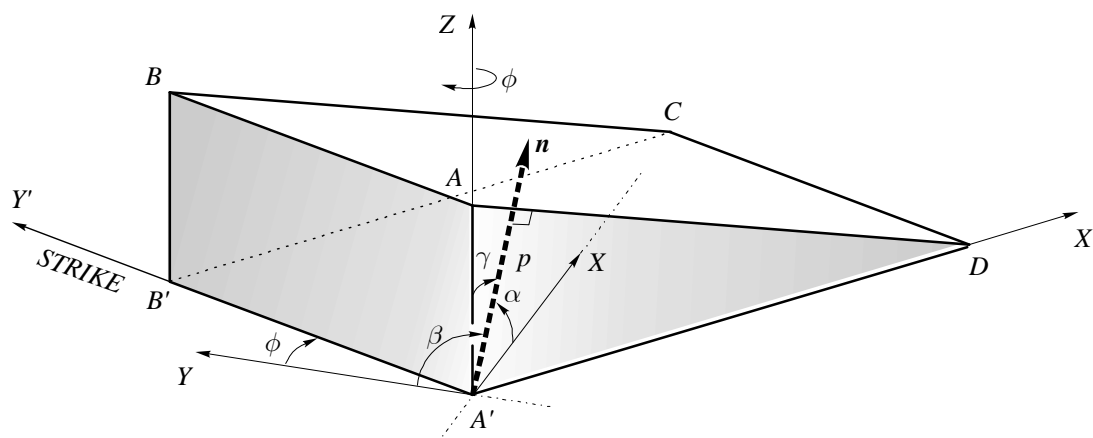


Figure 3.1

In Figure 3.1, $ABCD$ is a portion of an inclined plane. A' and B' are vertical projections of A and B onto a horizontal X - Y plane ($A'B'CD$) and the line CD is the intersection of the inclined and horizontal planes. The XYZ Cartesian coordinate origin is at A' with the Z -axis vertical.

The equation of the inclined plane can be expressed as

$$lX + mY + nZ = p \quad (3.1)$$

where

$$\begin{aligned} l &= \cos \alpha \\ m &= \cos \beta \\ n &= \cos \gamma \end{aligned} \quad (3.2)$$

are known as *direction cosines* and p is the perpendicular distance from the coordinate origin to the plane. In the diagram, \mathbf{n} is the normal to the plane and is shown as the thick dotted line, which lies in the plane $AA'D$.

This equation is known as the *normal* equation of a plane; ie the plane is defined by fixing the direction of the normal \mathbf{n} to the plane by means of the three angles α , β and γ . These are the angles between the X , Y and Z coordinate axis respectively and the normal.

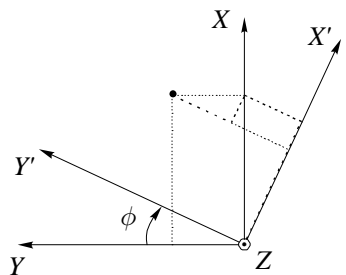
The direction cosines l , m and n have the property

$$l^2 + m^2 + n^2 = 1 \quad (3.3)$$

3.2 Direction of Strike and Maximum Dip on an Inclined Plane

In Figure 3.1, the line AB is a level line on the inclined plane and is known as the *strike line*. CD , which is parallel to AB , is also a strike line as is any other parallel line in the inclined plane. The line perpendicular to the strike line is the direction of *maximum dip*. In Figure 1 the Y' -axis is the direction of strike and the X' -axis is the direction of maximum dip.

The direction of strike can be determined from the normal equation of the plane by considering a clockwise rotation of the X - Y axes about the Z -axis by an angle ϕ . If the Y -axis is the direction of north then ϕ will be the bearing of the strike line of the inclined plane. A clockwise rotation about the Z -axis can be represented by the matrix equation



$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad (3.4)$$

Referring to Figure 3.1, when the Y' -axis is the direction of strike, the Y' coordinate of any point along the normal to the inclined plane will be zero, ie, the normal will lie in the Z - X' plane. Hence, from equation (3.4)

$$Y' = X \sin \phi + Y \cos \phi = 0 \quad (3.5)$$

Now the X and Y coordinates of the point where the normal pierces the inclined plane are $p \cos \alpha$ and $p \cos \beta$ respectively, giving

$$\tan \phi = \frac{-p \cos \beta}{p \cos \alpha} = \frac{-m}{l} \quad (3.6)$$

Note that the "whole circle" bearing ϕ ($0^\circ \leq \phi < 360^\circ$) must be determined by resolving the correct quadrant for the angle ϕ given by equation (3.6).

3.2 Intersecting Batter Planes

In Figure 3.2, $HBAJ$ and $ABKL$ are batter planes of slopes 1 in s_1 and 1 in s_2 which intersect along the line AB . For volume computation and setting-out purposes, it is often required to determine the direction and slope of the intersection line AB .

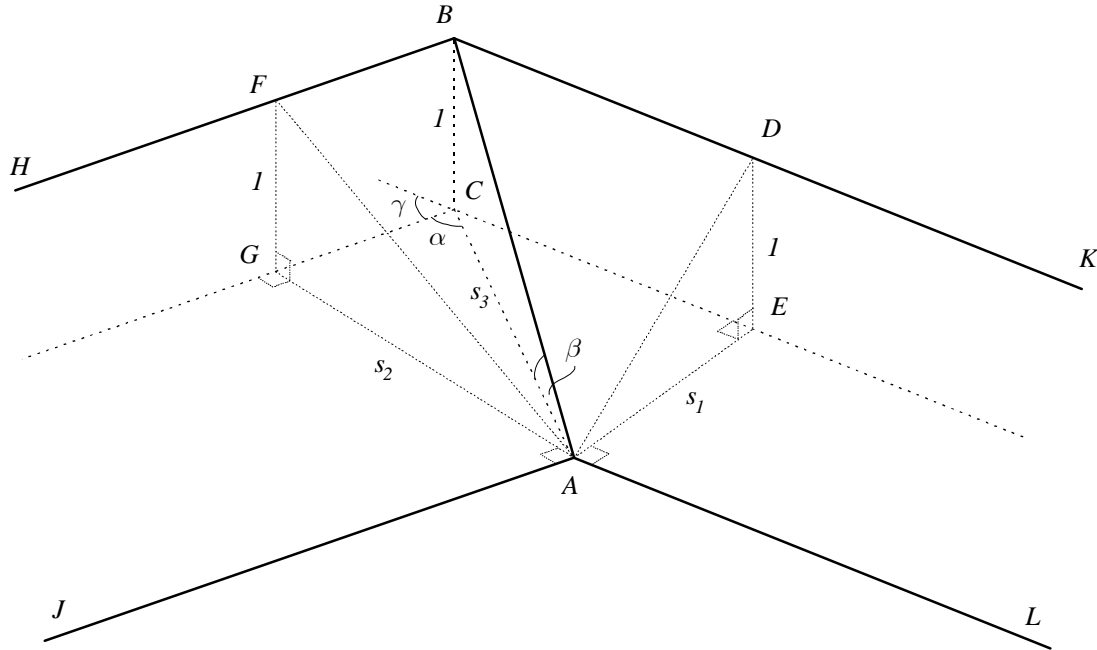


Figure 3.2 Batter Planes intersecting at angle γ

The slope of the batter plane $ABKL$ is 1 in s_1 shown diagrammatically by the right-angle triangle ADE . Similarly, the slope of the batter plane $HBAJ$ is 1 in s_2 shown diagrammatically by the right-angle triangle AFG . The angle between the lines AJ and AL (or BH and BK) is γ . This is also the angle between the lines BH and BK and CG and CE . The angle between the triangular plane BCA and the vertical plane $FBCG$ is α . This is the direction of the line of intersection of the two inclined batter planes.

Given s_1 and s_2 for the two batter planes and the angle γ between the strike lines (or level lines) of the two planes, the slope and direction of the line of intersection can be found in the following manner.

In right angle triangle AGC

$$s_3 = \frac{s_2}{\sin \alpha} \quad (3.7)$$

In right angle triangle ACE

$$s_3 = \frac{s_1}{\sin(180 - (\gamma + \alpha))} = \frac{s_1}{\sin(\gamma + \alpha)} \quad (3.8)$$

Equating (3.7) and (3.8) gives

$$\begin{aligned} s_1 \sin \alpha &= s_2 \sin(\gamma + \alpha) \\ &= s_2 (\sin \gamma \cos \alpha + \sin \alpha \cos \gamma) \end{aligned}$$

Re-arranging gives

$$\sin \alpha (s_1 - s_2 \cos \gamma) = s_2 \sin \gamma \cos \alpha \quad (3.9)$$

Dividing both sides of equation (3.9) by $\sin \alpha$ and re-arranging gives

$$\tan \alpha = \frac{s_2 \sin \gamma}{s_1 - s_2 \cos \gamma} \quad (3.10)$$

The slope of the line of intersection is 1 in s_3 and having calculated α from equation (3.10), s_3 can be obtained from equation (3.7). The angle of elevation β of the line of intersection is given by

$$\tan \beta = \frac{1}{s_3} = \frac{\sin \alpha}{s_2} \quad (3.11)$$

Note, that when $\gamma = 90^\circ$, see Figure 3.3 $\tan \alpha = \frac{s_2}{s_1}$ and $s_3 = \sqrt{s_1^2 + s_2^2}$. Furthermore, when $\gamma = 90^\circ$ and $s_1 = s_2 = s$ then $\alpha = 45^\circ$ and $s_3 = s\sqrt{2}$.

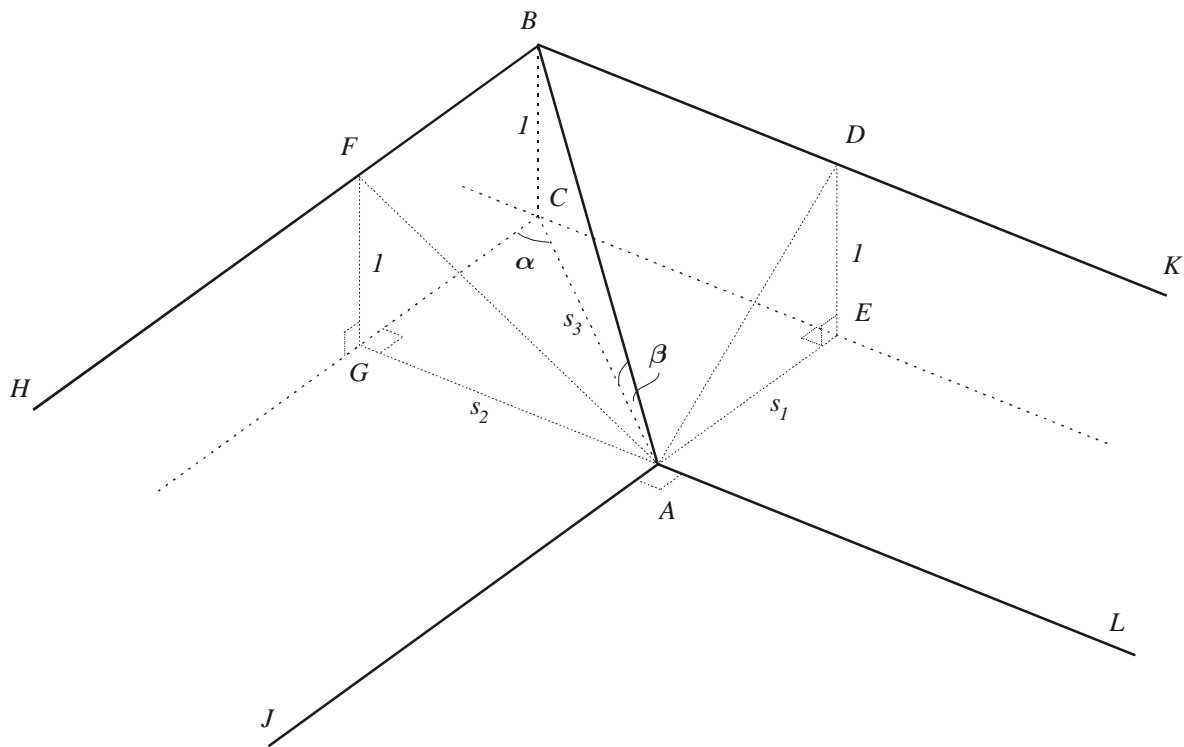


Figure 3.3 Batter Planes intersecting at right angles

3.3 Convergent Grades

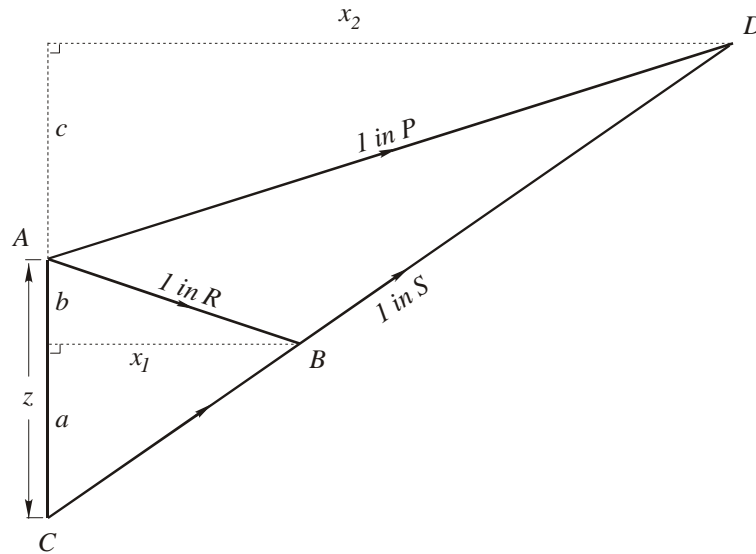


Figure 3.4 Convergent grades

In Figure 3.4 AC is a vertical line, z is the vertical height difference between A and C and the rising grades 1 in P (1 vertical to P horizontal) and 1 in S intersect at D . The grades 1 in R (falling grade) and 1 in S (rising grade) intersect at B . CBD is a straight line, B is a horizontal distance x_1 from AC and D is a horizontal distance x_2 from AC .

Given the grades 1 in S (rising from C) and 1 in R (falling from A) intersecting at B and the vertical height difference z between A and C the horizontal distance x_1 can be determined as follows.

$$\frac{1}{S} = \frac{a}{x_1} \quad (3.12)$$

$$\frac{1}{R} = \frac{b}{x_1} \quad (3.13)$$

adding equations (3.12) and (3.13) noting that $z = a + b$ gives

$$\frac{1}{S} + \frac{1}{R} = \frac{a}{x_1} + \frac{b}{x_1} = \frac{z}{x_1}$$

re-arranging gives

$$x_1 = \frac{z(R \times S)}{R + S} \quad (3.14)$$

Given the grades 1 in S (rising from C) and 1 in P (rising from A) intersecting at D and the vertical height difference z between A and C the horizontal distance x_2 can be determined as follows.

$$\frac{1}{S} = \frac{z+c}{x_2} = \frac{z}{x_2} + \frac{c}{x_2} \quad (3.15)$$

$$\frac{1}{P} = \frac{c}{x_2} \quad (3.16)$$

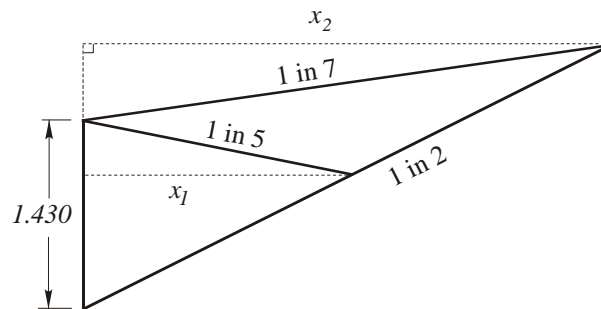
Substituting (3.16) into (3.15) gives

$$\frac{1}{S} = \frac{z}{x_2} + \frac{1}{P}$$

re-arranging gives

$$x_2 = \frac{z(P \times S)}{P - S} \quad (3.17)$$

Example: In the diagram below, calculate the horizontal distances x_1 and x_2



$$x_1 = \frac{1.430(5 \times 2)}{5 + 2} = 2.043$$

$$x_2 = \frac{1.430(7 \times 2)}{7 - 2} = 4.004$$

3.4 Natural Surface Plane, Intersecting Batter Planes and Convergent Grades Example

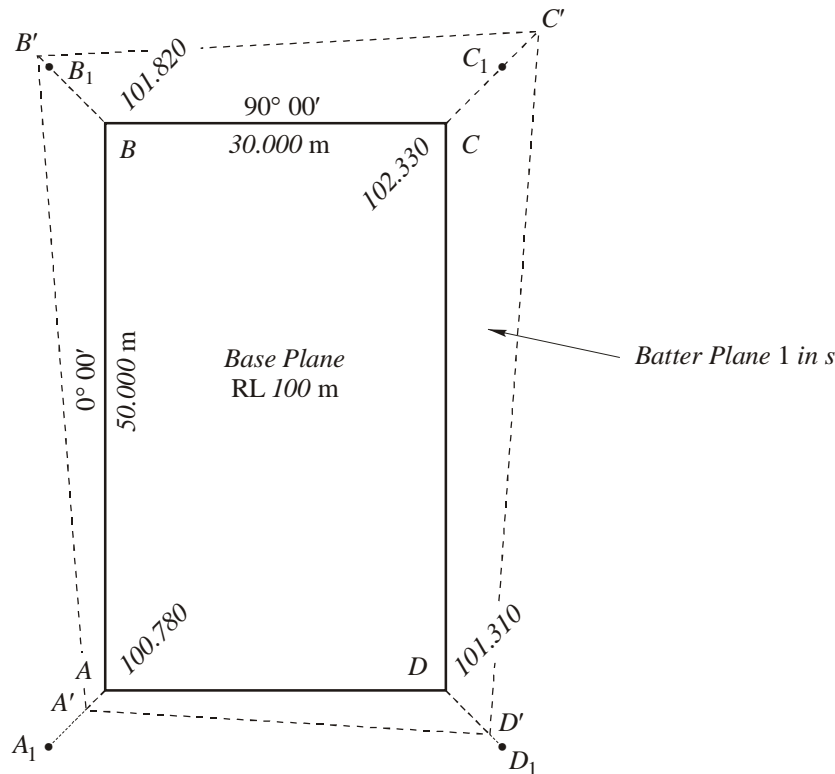


Figure 3.5

Figure 3.5 is a plan view of a proposed rectangular excavation on sloping ground. Soil is to be excavated to a rectangular level plane $ABCD$ having a reduced level (RL) of 100 m. The excavation is to have batter planes of 1 in s (1 vertical to s horizontal) where $s = 3$. The dotted line $A'B'C'D'$ is the extent of the excavation, i.e., where the batter planes intersect the natural surface. The rectangle $ABCD$ (50 m by 30 m) has been marked on the ground and RL's of the corners are shown.

It is required to calculate the positions of the limits of excavation, i.e., the locations of A' , B' , C' and D' .

- Method of solution:
- Assuming that the natural surface is a plane, use the RL's of A , B and C to compute the *normal* equation of the natural surface plane. Check that the computed RL of D (from the equation of the plane) is close to the measured natural surface RL.
 - Calculate the direction and slope (1 in s_3) of the lines of intersection of the batter planes, i.e., the direction and slope of the batter plane intersection lines AA' , BB' , CC' and DD' .
 - Calculate the RL's of points on the natural surface plane near A' , B' , C' and D' in the direction of the lines of intersection of the batter planes. These points are denoted A_1 , B_1 , C_1 and D_1 . Use these RL's and the RL's of A , B , C and D to determine natural surface grades (1 in n) along the lines of intersection.
 - Use the formula for convergent grades to calculate the horizontal distances AA' , BB' , CC' and DD' .

Solution

Part (i) Assign X, Y coordinates of 100.0 m and 100.0 m to A and treat RL's as Z -coordinates

Point	X	Y	Z
A	100.000	100.000	100.780
B	100.000	150.000	101.820
C	130.000	150.000	102.330
D	130.000	100.000	101.310

Calculate the components of two vectors $\mathbf{a} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{BC}$. These two vectors define the natural surface plane.

$$\begin{aligned}\mathbf{a} &= (X_A - X_B)\mathbf{i} + (Y_A - Y_B)\mathbf{j} + (Z_A - Z_B)\mathbf{k} \\ &= 0\mathbf{i} - 50\mathbf{j} - 1.040\mathbf{k} \\ \mathbf{b} &= (X_C - X_B)\mathbf{i} + (Y_C - Y_B)\mathbf{j} + (Z_C - Z_B)\mathbf{k} \\ &= 30\mathbf{i} + 0\mathbf{j} + 0.510\mathbf{k}\end{aligned}$$

Calculate the normal to the plane using the vector cross product $\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{p}} = \mathbf{p}$. The perpendicular vector \mathbf{p} will be in the direction of the "upward" normal to the natural surface plane.

$$\begin{aligned}\mathbf{p} = \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -50 & -1.040 \\ 30 & 0 & 0.510 \end{vmatrix} = -25.500\mathbf{i} - 31.200\mathbf{j} + 1500.000\mathbf{k}\end{aligned}$$

The magnitude $|\mathbf{p}| = 1500.541132$ and the unit vector $\hat{\mathbf{p}} = -0.016994\mathbf{i} - 0.020792\mathbf{j} + 0.999639\mathbf{k}$. The components of the unit vector $\hat{\mathbf{p}}$ are the direction cosines l , m and n .

$$\begin{aligned}l &= -0.016994 \\ m &= -0.020792 \\ n &= 0.999639\end{aligned}$$

The *normal* equation of the natural surface plane is

$$lX + mY + nZ = p$$

where p is the perpendicular distance from the origin to the plane. The value of p may be determined by substituting the coordinates of A into the equation of the plane

$$p = lX_A + mY_A + nZ_A = 96.965019$$

The computed RL (from the natural surface plane) is given by re-arranging the equation for the plane

$$Z_D = \frac{p - (lX_D + mY_D)}{n} = 101.290 \text{ m}$$

This value agrees closely (0.020 m) with the observed RL of D .

Part (ii) The batter plane slopes are 1 in s where $s = 3$. Since the excavation is rectangular, the batter planes intersect at right angles and their grades are all the same, hence the batter intersection lines AA' , BB' , CC' and DD' will be at angles of 45° to the rectangular plane $ABCD$ with grades of 1 in s_3 where

$$s_3 = s\sqrt{2} = 3\sqrt{2}$$

Part (iii) Calculate the RL's of points near A' , B' , C' and D' in the directions of the batter intersection lines. Call these points A_1 , B_1 , C_1 and D_1 and assign coordinates ± 5 m from the corners A , B , C and D to these points and calculate the RL's from the normal equation of the plane. This will mean that A_1 , B_1 , C_1 and D_1 will be located at bearings 225° , 315° , 45° and 135° respectively at distances $\sqrt{50} = 7.071$ m from the corners A , B , C and D .

Normal equation of plane $lX + mY + nZ = p$

$$l = -0.016994$$

$$m = -0.020792$$

$$n = 0.999639$$

$$p = 96.965019$$

Point	X	Y	Z (computed)
A_1	95.000	95.000	100.591
B_1	95.000	155.000	101.839
C_1	135.000	155.000	102.519
D_1	135.000	95.000	101.271

Part (iv) Calculate the natural surface grades (1 in n) in the direction of the batter intersection lines using the computed Z -values at A_1 , B_1 , C_1 and D_1 and the observed RL's of A , B , C and D .

Line	$dist$	ΔZ	grade 1 in n
$A - A_1$	7.071	$100.591 - 100.780 = -0.811$	$n = -8.719$
$B - B_1$	7.071	$101.839 - 101.820 = 0.019$	372.216
$C - C_1$	7.071	$102.519 - 102.330 = 0.189$	37.413
$D - D_1$	7.071	$101.271 - 101.310 = -0.039$	-181.308

Calculate the horizontal distances d from A , B , C and D to A' , B' , C' and D' respectively using the formula for convergent grades. Note that falling grades are shown with a negative sign.

Line	grade 1 in n	grade 1 in $s_3 s$	z (depth of cut)	$d = \frac{z(n \times s_3)}{n - s_3}$
$A - A'$	-8.719	$3\sqrt{2} = 4.243$	0.780	2.226
$B - B'$	372.216	$3\sqrt{2} = 4.243$	1.820	7.811
$C - C'$	37.413	$3\sqrt{2} = 4.243$	2.330	11.151
$D - D'$	-181.308	$3\sqrt{2} = 4.243$	1.310	5.431